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RESPONSE OF A SIMPLY-SUPPORTED TIMOSHENKO
BEAM TO A PURELY RANDOM GAUSSIAN PROCESS

by

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to

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by

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<u>ABSTRACT</u>

The generalized Fourier analysis is applied to the damped
Timoshenko beam equation to calculate the mean square values of dis-
placements and bending stress, resulting from purely random loading.
Compared with the calculations based on the classical beam theory [4],
it was found that the displacement correlations of both theories were
in excellent agreement. Moreover the mean square of the bending
stress, contrary to the results of the classical beam theory, was found
to be convergent. Computations carried out with a digital computer
are plotted for both theories.

1. Introduction

The response of linear elastic beams to random forcing functions has been studied by Ornstein [1], Houdijk [2], Van Lear and Uhlenbeck [3], and Eringen [4]. Eringen calculated the mean square displacements as well as the mean square stresses produced in a beam subject to a purely random loading, and found that the series for the mean square stresses diverge. He then suggested that the cause of this divergence may be traced to the inadequateness of the classical beam theory. In order to ascertain this point, one is then led to consider a more refined theory containing a mechanism which will favorably respond to such a loading. The present study is the result of such a consideration.

At first, the Timoshenko beam theory modified with the addition of a translatory velocity damping appeared to be adequate. However, a close examination showed that not only this, but the addition of Voigt-Sezawa type of internal damping did not produce an adequate theory leading to converging series for the mean square bending stress. Introduction of a linear damping to rotatory motion, however, produced the expected result. With the addition of such a mechanism to Timoshenko beam, it was no longer necessary to complicate the analysis by using internal damping. The analysis of this model is carried out for a simply supported beam subject to random pressure and purely random concentrated load.

An electronic computer was used to compute the displacement and bending stress correlation functions.
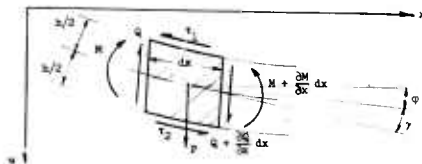
2. Differential Equations

Fig. 1 Element of the Beam

Equations of rotatory and translatory motions of an element of the Timoshenko beam respectively are (Fig. 1)

$$(1) \quad \frac{\partial M}{\partial x} + Q - \frac{h}{2} (\tau_1 + \tau_2) = \rho I \frac{\partial^2 \varphi}{\partial t^2}$$

$$(2) \quad \frac{\partial Q}{\partial x} + P + (\tau_2 - \tau_1) \frac{\partial w}{\partial x} = \rho A \frac{\partial^2 w}{\partial t^2} + \beta_0 \frac{\partial w}{\partial t}$$

where M, Q, P are the bending moment, vertical shearing force and the vertical applied load; τ_1, τ_2 are surface shearing stresses which will play the role of a damping due to rotation; h, A, I are the thickness, the cross section area, and the moment of inertia about the neutral axis, ρ is the mass density, φ and w are the bending angle and the deflection. Coordinate x locates the cross section and t is the time.

In the Timoshenko beam theory we have

(3) $\frac{\partial w}{\partial x} = \varphi + \gamma, \quad \varphi \cong \frac{\partial w_B}{\partial x}$

(4) $M = -EI \frac{\partial \varphi}{\partial x} \cong -EI \frac{\partial^2 w_B}{\partial x^2}$

(5) $Q = kGA \gamma$

Where γ is the shearing angle, E and G are Young's and shear modulii respectively, and k is a constant which is adjustable to take into accound the effect of the shape of the cross section. Here w_B is termed the bending deflection [5]. For k, Timoshenko and E. Reissner respectively use 2/3 and 5/6.

In order to arrive at a rotatory damping similar to linear velocity damping, we set

(6) $\tau_1 = \tau_2 = \frac{1}{h} \beta_1 \frac{\partial \varphi}{\partial t}$, β_1 constant.

This merely introduces a mechanism for the rotatory damping and is no more or less a sacrifice then the universally accepted idea of linear damping $\beta_0 \, \partial w/\partial t$ present in (2). It may be thought of as being the air friction combined with all the frictional resistance against the rotation of the element of the beam.

If we use (3) - (6) in (1) and (2), we obtain:

(7) $EI \frac{\partial^2 \varphi}{\partial x^2} - kGA\varphi + kGA \frac{\partial w}{\partial x} = \rho I \frac{\partial^2 w}{\partial t^2} + \beta_1 \frac{\partial \varphi}{\partial t}$

$kGA \frac{\partial \varphi}{\partial x} = P + kGA \frac{\partial^2 w}{\partial x^2} - \beta_0 \frac{\partial w}{\partial t}$

Differentiating the first of (7) with respect to x and using the second of (7) to eliminate φ, we find

(8) $\quad Lw = P + \dfrac{1}{kGA} \left(-EI \dfrac{\partial^2 P}{\partial x^2} + \rho I \dfrac{\partial^2 P}{\partial t^2} + \beta_1 \dfrac{\partial P}{\partial t} \right)$

where

$$Lw = EI \dfrac{\partial^4 w}{\partial x^4} - \left(\rho I + \dfrac{\rho EI}{kG} \right) \dfrac{\partial^4 w}{\partial x^2 \partial t^2} - \left(\dfrac{\rho_0 EI}{kGA} + \beta_1 \right) \dfrac{\partial^3 w}{\partial x^2 \partial t}$$

(9) $\quad + \dfrac{\rho^2 I}{kG} \dfrac{\partial^4 w}{\partial t^4} + \left(\dfrac{\rho I \beta_0}{kGA} + \dfrac{\rho \beta_1}{kG} \right) \dfrac{\partial^3 w}{\partial t^3} + \left(\rho A + \dfrac{\beta_0 \beta_1}{kGA} \right) \dfrac{\partial^2 w}{\partial t^2} + \beta_0 \dfrac{\partial w}{\partial t}$

A simple relation exists between w and w_B. This is found by combining (2) - (5), integrating the result with respect to x and excluding a rigid body displacement from the result. Hence

(10) $\quad w_s = w - w_B = \dfrac{1}{kGA} \left(\rho I \dfrac{\partial^2 w_B}{\partial t^2} + \beta_1 \dfrac{\partial w_B}{\partial t} - EI \dfrac{\partial^2 w_B}{\partial x^2} \right)$

Sometimes w_s is called the shear deflection [5].
With the introduction of (10), equation (9) may be simplified to

(11) $\quad Lw_B = P$

Equation (8) is the Timoshenko beam equation modified with linear and rotatory damping and external loading. In what follows, we also need the boundary conditions for a simply-supported beam:

(12) $\quad w = \partial\varphi/\partial x = 0 \quad$ at $x = 0, L$

From (10), (3), and (4), it is clear that if we make

(13) $\quad w_B = \partial^2 w_B/\partial x^2 = 0 \quad$ at $x = 0, L$

then the boundary conditions (12) would be satisfied. Hence we may first solve (11) subject to (13) to obtain w_B. Equation (10) then gives w.

3. Formal Solution

In order to determine the mean square values of various quantities such as the displacements and the stresses, we must first find a steady-state solution of (8) satisfying (12). This is the same thing as solving (11) subject to (12). Generalized harmonic analysis [6] can then be used to complete the problem. A formal solution of (11) satisfying (12) has the form

$$(14) \quad w_B(x,t) = \sum_{n=1}^{\infty} w_n(t) \sin(n\pi x/L)$$

If we use (14) and a similar expression for P with $P_n(t)$ replacing $w_n(t)$ in (11), we obtain an ordinary differential equation for $w_n(t)$. The Fourier transform of this equation with respect to t gives:

$$(15) \quad \bar{w}_n(\zeta) = (c^4 \lambda / EI) \, \bar{P}_n(\zeta) / D_n(\zeta)$$

where

$$(16) \quad D_n(\zeta) = \zeta^4 - ia\zeta^3 - b_n\zeta^2 + ic_n\zeta + d_n$$

and

$$(17) \quad a = \beta_o A_o, \; b_n = B_n + \beta_o^2 B, \; c_n = \beta_o C_n, \; d_n = \lambda \lambda_n^4 c^4$$

$$A_o = (1 + \kappa)/\rho A, \; B = \kappa(\rho A)^{-2}, \; B_n = c^2[(1 + \lambda)\lambda_n^2 + \lambda r^{-2}]$$

$$(18) \quad C_n = (c^2/\rho A) \, [(1 + \kappa\lambda)\lambda^2 + \lambda r^{-2}], \; \lambda = kG/E, \; c = (E/\rho)^{1/2}$$

$$r = (I/A)^{1/2}, \; \kappa = \beta_1/\beta_o r^2, \; \lambda_n = n\pi/L$$

Further a bar on top of a letter represents the Fourier transform, e. g.,

$$(19) \quad \bar{v}_n(\zeta) = \int_{-\infty}^{\infty} v_n(t) \, e^{i\zeta t} dt$$

From the expression of $P(x,t)$ of the form (14), the Fourier coefficient $\bar{F}_n(\zeta)$ may be found to be

$$(20) \quad \bar{F}_n(\zeta) = \frac{2}{L} \int_0^L \bar{F}(\xi,\zeta) \sin(n\pi\xi/L) \, d\xi$$

With the use of (20) in (15) we may invert (15) to obtain

$$(21) \quad w_n(t) = (2c^4\lambda/\pi IL) \int_0^L [\int_{-\infty}^{\infty} W_n(t-\tau) \, P(\xi,\tau) \, d\tau] \sin(n\pi\xi/L) \, d\xi$$

where

$$(22) \quad W_n(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{it\zeta}/D_n(\zeta) \, d\zeta$$

is the weighting function of the system for the nth mode of vibration.

Through equation (14) we now have $w_n(x,t)$. If we further use this result in (10), we obtain $w(x,t)$, thus completing a steady-state solution of (8) subject to (12).

4. Evaluation of $W_n(t)$

Contour integration may conveniently be used to evaluate (21). For $t>0$ the appropriate contour C for this integral is the upper half of a circle of radius R in the ζ-plane having the origin at $\zeta = 0$. It can be shown that the integral

$$W_n(t) = \frac{1}{2\pi} \oint_C e^{it\zeta}/D_n(\zeta) \, d\zeta$$

along the semi-circular arc vanishes as $R \to \infty$, and that along the real axis gives (22), which by the theory of residues will be $2\pi i$ times the sum of the residues within the contour. To obtain the residues, we need to locate the poles of $D_n(\zeta) = 0$, which in view of (16) is the same thing as finding the roots of the frequency equation:

(23) $z^4 + sz^3 + b_n z^2 + C_n z + d_n = 0$, $z = i\zeta$

The roots of the quartic (23) can be determined exactly by using the known formulae for roots. However, the complexity of these formulae would make the result difficult to interpret. Thus we use an approximate method which, for small damping, introduces negligible errors in the final results. We take the transverse and rotatory damping coefficients β_o and β_1 to be the same order of magnitude. This means $\kappa = O(1)$. For β_o small a natural method for finding the roots of (16) is to use a perturbation procedure in which β_o is the perturbation parameter. Thus we write

(24) $z = z_o + \beta_o z_1 + \beta_o^2 z_2 + \ldots$

Upon substitution of this into (23), using (17) and setting the coefficients of various powers of β_o equal to zero, we obtain

$1 \cdot \cdot$ $z_o^4 + B_n z_o^2 + d_n = 0$

$\beta_o \cdot \cdot$ $4z_o^3 z_1 + 2z_o z_1 B_n + A_o z_o^3 + C_n z_o = 0$

$\cdot \cdot \cdot$

These equations may be solved very simply since z_0 satisfies a biquadratic while all the other z_i satisfy linear equations. Hence

$$z_0 = i\Omega_n \quad ; \quad \Omega_n = \omega_n, -\omega_n, \mu_n, -\mu_n ,$$

$$(25) \quad \omega_n = [\tfrac{1}{2}B_n + \tfrac{1}{2}(B_n^2 - 4d_n)^{\frac{1}{2}}]^{\frac{1}{2}}$$

$$\mu_n = [\tfrac{1}{2}B_n - \tfrac{1}{2}(B_n^2 - 4d_n)^{\frac{1}{2}}]^{\frac{1}{2}}$$

and

$$(26) \quad z_1 = -(A_0 z_0^2 + C_n)(4z_0^2 + 2B_n)^{-1}$$

We likewise obtain $z_2, \ldots$. However in the present work, it will not be necessary to obtain the higher order terms.

We note that the dependence of C_n on n does not violate the perturbation since for large n we have $z = O(n)$, thus making $C_n z$ and sz^3 $O(n^3)$ while the remaining terms are $O(n^4)$.

An examination will show that $B_n^2 - 4d_n > 0$. Hence ω_n and μ_n are real. Using (24) - (26) we may now write $D_n(\zeta)$ to a linear approximation in β_0 as

$$D_n(\zeta) \doteq (\zeta-\zeta_1)(\zeta + \zeta_1^*)(\zeta - \zeta_2)(\zeta + \zeta_2^*),$$

$$\zeta_1 = \omega_n + i\delta_n \quad , \quad \zeta_2 = \mu_n + i\alpha_n,$$

$$(27) \quad \delta_n = \beta_0(A_0\omega_n^2 - C_n)(4\omega_n^2 - 2B_n)^{-1}$$

$$\alpha_n = \beta_0(A_0\mu_n^2 - C_n)(4\mu_n^2 - 2B_n)^{-1}$$

and an asterisk represents complex conjugate e.g. $\zeta_1{}^* = \omega_n - i\delta_n$.

By use of Cauchy's theorem of residues, we find that $W_n(t) = 0$ for $t<0$ and

$$W_n(t) = \text{Im} \left(P_1 e^{i\zeta_1 t} + P_2 e^{i\zeta_2 t} \right) \quad \text{for } t>0$$

$$(28) \quad P_1 = \omega_n{}^{-1}(\zeta_1 - \zeta_2)^{-1}(\zeta_1 + \zeta_2{}^*)^{-1}$$

$$P_2 = \mu_n{}^{-1}(\zeta_2 - \zeta_1)^{-1}(\zeta_2 + \zeta_1{}^*)^{-1}$$

Here Im represents the imaginary part of the complex quantity. Substituting the first of (28) into (21) and the result into (14), we find

$$(29) \quad w_B(x,t) = -(2c^4\lambda/\text{EIL}) \sum_{n=1}^{\infty} \{\int_0^L \{\int_0^{\infty} \text{Im}(P_1 e^{i\zeta_1(t-\tau)} + P_2 e^{i\zeta_2(t-\tau)})$$

$$P(\xi,\tau)d\tau\} \sin (n\pi\xi/L)d\xi\} \sin (n\pi x/L)$$

This completes the solution of the deterministic problem. For through (10), we can calculate $v(x,t)$.

$$(30) \quad v(x,t) = -(2c^4\lambda/\text{EIL}) \sum_{n=1}^{\infty} \int_0^L \{\int_0^{\infty} \text{Im}\left(Q_1 e^{i\zeta_1(t-\tau)} + Q_2 e^{i\zeta_2(t-\tau)}\right)$$

$$P(\xi,\tau)d\tau\} \sin (n\pi\xi/L) \, d\xi\} \sin (n\pi x/L) ,$$

$$Q_j = [1 - (r^2/\lambda c^2)\zeta_j{}^2 + [\kappa\alpha r^2 \lambda^{-1} c^{-2}(1+\kappa)^{-1}] \, i\zeta_j + (r^2\lambda_n{}^2/\lambda) \} P_j,$$

$$(J = 1,2)$$

5. The Autocorrelation Function of the Displacement and the Bending Moment

The time average of a function $v(x,t)$ is defined by

$$(31) \quad \langle v(x,t) \rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} v(t, t + \tau) \, d\tau.$$

For an _ergodic process_ this average is known to be equal to the expectation (ensemble average) $E\{w(x,t)\}$ of the function $w(x,t)$. If the origin selected for time t does not effect the expectation, i.e.

$$E\{x(x,t+\tau)\} = E\{w(x,t)\}$$

we say that the process is stationary. For a stationary ergodic process, therefore, we may write:

$$E\{w(w,t)\} = \langle w(x,t) \rangle = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} w(x,t)\, w(\xi, t+\tau) dt$$

(32)

$$E\{w(x,t)\, w(t+\tau)\} = R_w(x,\xi,\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} w(x,t)\, w(\xi, t+\tau)\, dt$$

Here R_w is called the autocorrelation function of $w(x,t)$. We note that the mean square value of $w(x,t)$ follows from the second of (31) by set-ting $\tau = 0$. The aim of the present paper is to determine the auto-correlation functions of the displacement w and that of the bending stress in terms of the autocorrelation function $R_p(x,\xi,\tau)$ of the applied load, i.e.

(33) $\quad R_p(x,\xi,\tau) = E\{P(x,t)\, P(\xi, t+\tau)\}$

This latter quantity is obtained from a record of the time history of the load $P(x,t)$. Later for this quantity, we shall select the fundamental case of _purely random process_ in time with temporal corre-lation in two space points, vanishing where the points are not coincident. More precisely

(34) $R_p(x,\xi,\tau) = D\delta(x-\xi)\,\delta(\tau)$

where δ is the Dirac delta function. This form is commonly used in the treatment of the Brownian motion, but it is doubtful if any real function could comply with it exactly.

Using (14) with $w_n(t)$ given by (21) in place of w in the second of (32), we obtain

(35) $R_{v_B} = (2c^4\lambda/EIL)^2 \sum\limits_{m,n=1}^{\infty} [\int_0^L \int_0^L [\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_m(r)\,W_n(s)$

$R_p(x',\xi',\tau+r-s)\,dr\,ds]\,dx'd\xi']\,\sin(n\pi\xi/L)\,\sin(m\pi\xi/L)$

Upon substitution of (34) this gives

(36) $R_{v_B} = \dfrac{D}{L}(c^4\lambda/EI)^2 \sum\limits_{n=1}^{\infty} S_n(\tau)\,\sin(n\pi x/L)\,\sin(n\pi\xi/L)$

where

(37) $S_n(\tau) = 2\int_{-\infty}^{\infty} W_n(r)\,W_n(r+\tau)\,dr$

We may now use (28) in (37) to evaluate this integral

$S_n(\tau) = \text{Real}\,(M_1 e^{i\zeta_1\tau} + M_2 e^{i\zeta_2\tau})$

where

(38) $M_1 = -i(2\zeta_1)^{-1}P_1^2 + i(\zeta_1 - \zeta_1{}^*)^{-1}P_1P_1{}^* - i(\zeta_1 + \zeta_2)^{-1}P_1P_2 +$

$\quad i(\zeta_1 - \zeta_2{}^*)^{-1}P_1P_2{}^*$

$\quad M_2 = -i(2\zeta_2)^{-1}P_2^2 + i(\zeta_2 - \zeta_2{}^*)^{-1}P_2P_2{}^* - i(\zeta_2 + \zeta_1)^{-1}P_2P_1 +$

$\quad i(\zeta_2 - \zeta_1{}^*)^{-1}P_2P_1{}^*$

The real form of (38) is

$$S_n(\tau) = e^{-b_n \tau}(E_n \cos \omega_n \tau + F_n \sin \omega_n \tau)$$

$$+ e^{-\alpha_n \tau}(G_n \cos \mu_n \tau + H_n \sin \mu_n \tau)$$

(39)

$$2E_n = M_1 + M_1^* \qquad , \qquad 2F_n = i(M_1 - M_1^*)$$

$$2G_n = M_2 + M_2^* \qquad , \qquad 2H_n = i(M_2 - M_2^*)$$

For small β_0 these may be simplified to

$$E_n \cong (2b_n)^{-1} \omega_n^{-\omega}(\omega_L^\omega - \mu_n^2)^{-2}$$

(40) $$F_n \cong (2\omega_n)^{-1} \omega_n^{-2}(\omega_n^2 - \mu_n^2)^{-2} + 2\omega_n^{-1}(\omega_n^2 - \mu_n^2)^{-3}$$

$$G_n \cong (2\alpha_n)^{-1} \mu_n^{-2}(\omega_n^2 - \mu_n^2)^{-2}$$

$$H_n \cong (2\mu_n)^{-1} \mu_n^{-2}(\omega_n^2 - \mu_n^2)^{-2} - 2\mu_n^{-1}(\omega_n^2 - \mu_n^2)^{-3}$$

The total displacement correlation R_w is obtained by substituting the first of (30) into the second of (31). The result is identical in form to (36) and (38) with M_1 and M_2 calculated by replacing P_1 and P_2 by Q_1 and Q_2 respectively. The real form of R_w may be written as

(41) $$R_w = \frac{D}{L}(\frac{c^4\lambda}{EI})^2 \sum_{n=1}^{\infty}(e^{-b_n \tau}[E_n' \cos \omega_n \tau + F_n' \sin \omega_n \tau] + e^{-\alpha_n \tau}$$

$$[G_n' \cos \mu_n \tau + H_n' \sin \mu_n \tau]) \sin \frac{n\pi x}{L} \sin \frac{n\pi \xi}{L}$$

Where

$$2E_n' = M_1 + M_1^* \qquad , \qquad 2F_n' = i(M_1 - M_1^*)$$

$$2G_n' = M_2 + M_2^* \qquad , \qquad 2H_n' = i(M_2 - M_2^*),$$

which, for small β_0, simplify to

$$2E_n' \approx \delta_n^{-1} \omega_n^{-2}(\omega_n^2 - \mu_n^2)^{-2} q_{1n}^2,$$

$$2F_n' \approx \omega_n^{-3}(\omega_n^2 - \mu_n^2)^{-2} q_{1n}^2 + 4\omega_n^{-1}(\omega_n^2 - \mu_n^2)^{-3} q_{1n}q_{2n},$$

(42) $$2G_n' \approx \alpha_n^{-1} \mu_n^{-2}(\omega_n^2 - \mu_n^2)^{-2} q_{2n}^2,$$

$$2H_n' \approx \mu_n^{-3}(\omega_n^2 - \mu_n^2)^{-2} q_{2n}^2 + 4\mu_n^{-1}(\mu_n^2 - \omega_n^2)^{-3} q_{1n} q_{2n}$$

$$q_{1n} = 1 - \lambda c^{-2} r^2 \omega_n^2 + \lambda r^2 \lambda_n^2,$$

$$q_{2n} = 1 - \lambda c^{-2} r^2 \mu_n^2 + \lambda r^2 \lambda_n^2.$$

In order to calculate the correlation function for the bending stress, we first calculate the bending moment $M(x,t)$:

(43) $M(x,t) = -EI\, \partial^2 w_B/\partial x^2$

Thus the bending moment correlation R_M may be calculated from

$$R_M = E\{M(x,t)\, M(\xi, t + \tau)\} = (EI)^2 < \frac{\partial^2 w_B}{\partial x^2}\, \frac{\partial^2 w_B}{\partial \xi^2} > = (EI)^2\, \frac{\partial^4 R_{w_B}}{\partial x^2 \partial \xi^2}$$

When (35) is substituted, this gives

$$(44)\quad R_M = (Dc^8\lambda^2/L)\sum_{n=1}^{\infty} S_n(\tau)(n\pi/L)^4\, \sin\,(n\pi x/L)\, \sin\,(n\pi \xi/L)$$

An examination of $S_n(\tau)$ shows that it is $O(1/n^6)$. Hence

$$(n\pi/L)^4\, S_n(\tau) = O(1/n^2)$$

Therefore, we may conclude that (44) converges though possibly quite slowly. The maximum bending stress is given by $\sigma_x = M/Z$ where Z is the section modulus of the beam. The bending stress correlation would therefore be

$$(45)\quad R_{\sigma_x} = Z^{-2}E\{M(x,t)\, M(\xi, t + \tau)\} = (Dc^4\lambda^2/LZ^2)\sum_{n=1}^{\infty}(n\pi/L)^4\, S_n(\tau)$$

$$\sin\,(n\pi x/L)\, \sin\,(n\pi \xi/L)$$

6. Stress Correlations for Other Random Loadings

Two other special types of loadings are of practical importance. They are the purely random pressure and purely random concentrated load of which the first is represented by a pressure correlation of this form

(46) $\quad R_p = D_1 \, \delta(\tau)$

where D_1 is the constant spectral density of pressure $P(x,t)$. The maximum bending stress correlation in this case, calculated on the basis of the classical (Bernoulli-Euler) theory, is given by

(47) $\quad R_{\sigma_x} = (D_1 r^4 c^4 \pi^2 / 4 L^2 z^2) \sum\limits_{m,n=1,3,5..}^{\infty} mn \, T_{mn}(\tau) \sin (m x/L) \sin (n\pi \xi/L)$

where

$$T_{mn}(\tau) = \tfrac{1}{2} \omega_m^{-1} \omega_n^{-1} e^{-\delta\tau} [[4\delta^2 + (\omega_m - \omega_n)^2]^{-1} [2\delta \cos \omega_n \tau + (\omega_m - \omega_n)$$

$$\sin \omega_n \tau]$$

(48) $\quad -[4\delta^2 + (\omega_m + \omega_n)^2]^{-1} [2\delta \cos \omega_n \tau - (\omega_m + \omega_n) \sin \omega_n \tau]]$

$$\omega_n = (\alpha^{-1} \lambda_n^4 - \delta^2)^{\tfrac{1}{2}} \quad , \quad \alpha = \rho A / EI \quad , \quad \delta = \beta / 2\alpha \quad ,$$

$$\beta = \beta_0 / EI \quad , \quad \lambda_n = \tfrac{n\pi}{L} .$$

In the case of purely random concentrated load acting at $x = a$, the load correlation is

(49) $\quad R_p = D_2 \, \delta(x - a) \, \delta(\xi - a) \, \delta(\tau)$

The maximum bending stress correlation, based on the classical theory, is given in [4]. The results based on the Timoshenko model is as follows:

$$(50) \quad R_{\sigma_x} = (4D_2\lambda^2 c^8/L^2 z^2) \sum_{m,n=1}^{\infty} S_{mn}(\tau) \, (m\pi/L)^2 \, (n\pi/L)^2 \sin(m\pi a/L)$$

$$\sin(n\pi a/L) \cdot (m\pi x/L) \sin(n\pi \xi/L)$$

where

$$S_{mn}(\tau) = \int_{-\infty}^{\infty} W_m(r) \, W_n(r + \tau) \, dr$$

with W_n given by the first of (28). For other types of load correlations, see [4].

7. Numerical Example and Discussion

An electronic digital computer was used to calculate the displacement on maximum stress correlations at $x = \xi$ (the mean square values) for the case of purely random loading. The steel beam used was neither thick nor thin ($L/r = 20$) and had the following characteristics

$$E = 30 \times 10^6 \text{ psi} \qquad , \qquad G = 12 \times 10^6 \text{ psi} \qquad ,$$

$$k = 5/6 \qquad , \qquad \rho = 0.285 \text{ lb/in}^3 \qquad ,$$

$$L/r = 20 \qquad , \qquad \mu_1/2\pi = 151.5 \text{ cps} \qquad , \qquad \omega_1/\omega\pi = 3900 \text{ cps} \qquad ,$$

$$\kappa = 1.0$$

$$\alpha_n = \delta_n = 95.8; \ 700.$$

The series for R_{W_B} and R_W converge extremely rapidly, thus requiring

the calculations of only the first few terms. The series for R_{σ_x} is however very slowly convergent. For a fair accuracy, it was necessary to take into account 100 terms of the series. This, of course, was to be expected, since the Bernoulli-Euler Theory led to divergence for this quantity [4]. The results of these computations are plotted in Figures 2 - 5. For the purpose of comparison, the results of the Bernoulli-Euler Theory is also plotted in Figures 2 and 3. It is clear from these curves that Bernoulli-Euler beam theory is satisfactory for calculations of the mean square of the displacement. The curves of Figures 4 and 5 were, on the other hand, not obtainable from this theory. Figure 4 shows that the mean square of the bending stress is almost constant along the beam. According to Figure 5, this quantity, in time, is nearly purely random, as effectively indicated by the calculations based on the classical beam theory [4]. The damping has the effect of reducing the sharpness of the correlation at $\beta\tau = 0$.

In conclusion, the Bernoulli-Euler beam theory represents an adequate model for studying random vibrations of beams, when only the mean displacements are sought. If the mean bending stress is desired, it is necessary to use a more improved theory. The Timoshenko beam theory, as indicated by the present study, appears to be adequate for this purpose. The series obtained for the mean square bending stress is, however, slowly convergent. It would seem desirable to incorporate some form of internal damping to the theory so as to improve the convergence and to bring the theory into closer agreement with reality.
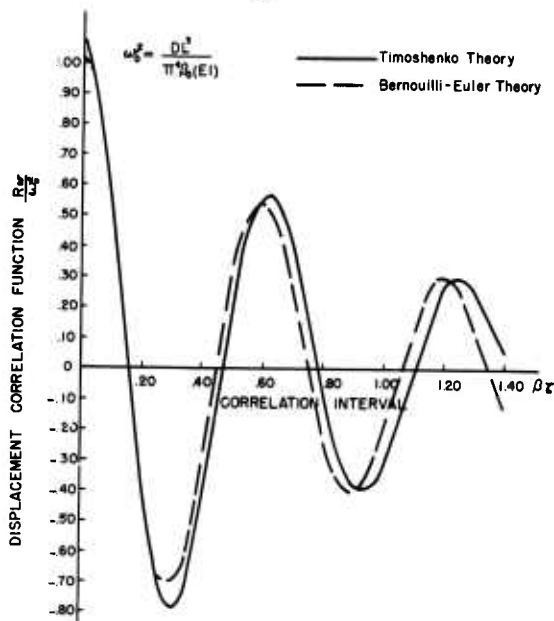
FIGURE 3. DISPLACEMENT CORRELATION
FUNCTION (Unnormalized) vs.
CORRELATION FUNCTION

FIGURE 4. MEAN SQUARE STRESS vs. POSITION ALONG BEAM

$R_\sigma = R_\sigma(0) = \dfrac{\overline{\sigma^2}}{\sigma^2}$

$\sigma_0^2 = \dfrac{DC_n^2 L}{2\pi \mu \beta^2 Z^4}$
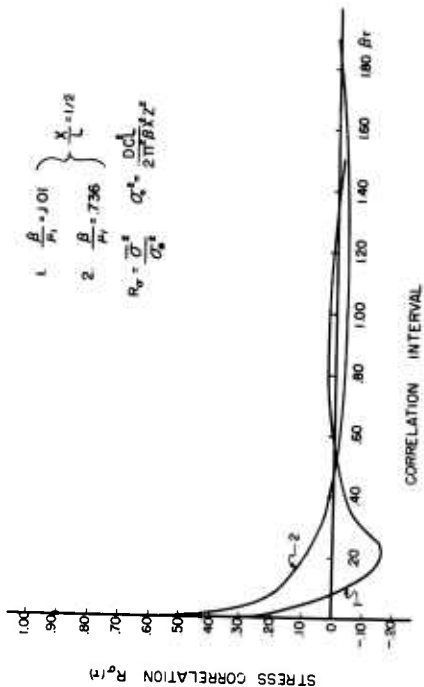
FIGURE 5. STRESS CORRELATION (Unnormalized) vs. CORRELATION INTERVAL

FIGURE 2. MEAN SQUARE DISPLACEMENT ALONG BEAM

- - - Timoshenko Theory
— — Bernoulli-Euler Theory

$$\omega^2 = \frac{DL^3}{\pi^2 A (EI)}$$

MEAN SQUARE DISPLACEMENT $\frac{u_m^2}{u_x^2}$

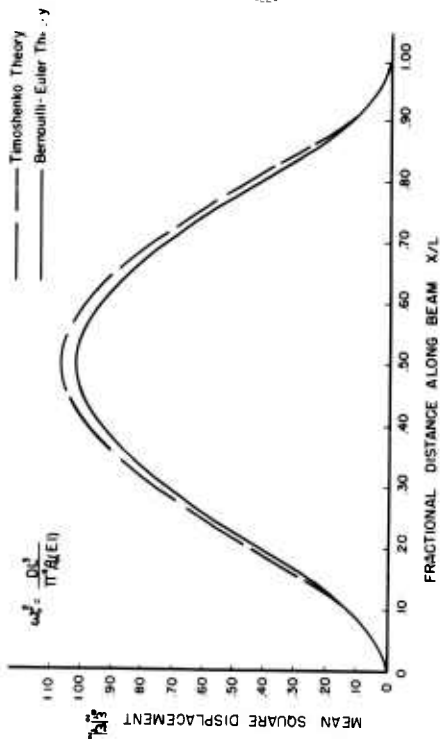FRACTIONAL DISTANCE ALONG BEAM X/L

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